Simple connectedness of quasitilted algebras

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Abstract

Let A be a basic connected finite dimensional algebra over an algebraically closed field. Assuming that A is quasitilted, we prove that A is simply connected if and only if $HH^1(A) = 0$. This generalises a result of I. Assem, F. U. Coelho and S. Trepode and which proves the same equivalence for tame quasitilted algebras.

Introduction

Let A be a finite dimensional algebra over an algebraically closed field k. In order to study the category $mod(A)$ of finite dimensional (right) A-modules we may assume that A is basic and connected. In this study, covering techniques introduced in [5] and [21] have proved to be a very powerful tool. Indeed, a Galois covering $\to A$ (with a locally bounded k-category) reduces the study of part of $mod(A)$ to the one of $mod()$ which is easier to handle (see for example [8]). From this point of view, simply connected algebras are of particular interest. Recall that if Q is the ordinary quiver of A and if kQ is the path algebra of Q, then there exists a (non necessarily unique) surjective algebra morphism (or presentation) $\nu\colon kQ \twoheadrightarrow A$ (see [4] for example). Moreover, given such a presentation, one can define the fundamental group $\pi_1(Q, Ker(\nu))$ of ν (see [20]). With this setting, the algebra A is called simply connected if and only if Q has not oriented cycle (*i.e.* no non trivial oriented path whose source equals its target, the algebra is then called triangular) and $\pi_1(Q, Ker(\nu)) = 1$ for any $\nu\colon kQ \twoheadrightarrow A$ (see [3]). Equivalently ([24] and [18]) A is simply connected if and only if there exists no Galois covering $\to A$ with non trivial group and with a connected locally bounded k-category.

To prove that A is simply connected seems to be a difficult problem, *a priori*, since one has to check that various groups are trivial. Hence, it is worth looking for a simpler characterisation of simple connectedness. It was asked by A. Skowroński ([25]) whether the equivalence "A is simply connected if and only if $HH^1(A) = 0$" is satisfied for A a tame triangular algebra. This equivalence is true for tilted algebras (see [2] for the tame case and [19] for the general case), for piecewise hereditary algebras of type any quiver (see [19]), for tame quasititled algebras (see [1]) and it is conjectured (*loc.cit.*) that this equivalence is true for any quasitilted algebra.

Recall ([12]) that a quasitilted algebra is an algebra isomorphic to $End_{\mathcal{H}}(T)^{op}$ where $\mathcal{H}$ is a hereditary abelian k-linear category (with finite dimensional Hom and Ext spaces) and where $T \in \mathcal{H}$ is a (basic) tilting object. In particular, a quasitilted algebra has global dimension at most 2 (see *loc.cit.*). Quasitilted algebras were introduced in order to give a common framework to the class of tilted algebras (introduced in [13]) and to the class of canonical algebras (introduced in [22]). In this text, we prove the following result:

Theorem 1. *Let A be a basic connected finite dimensional k-algebra. If A is quasitilted, then:*

$$A \text{ is simply connected } \Leftrightarrow\ HH^1(A) = 0$$

Moreover, if A is tilted of type Q, then A is simply connected if and only if Q is a tree.

Hence, the above theorem solves the above conjecture of [1] and it also answers positively the above question of A. Skowroński ([25]) for quasitilted algebras (of finite, tame or wild type). Recall that in Theorem 1, the case of tilted algebras and the one of quasitilted algebras which are derived equivalent

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to a hereditary algebra have been successfully treated in [2] and [19]. Here, we say hat two algebras are derived equivalent if and only if their derived categories of bounded complexes of finite dimensional modules are triangle equivalent.

In order to prove Theorem 1, we use ideas from [19]. More precisely, given a quasitilted algebra A which is not derived equivalent to a hereditary algebra, we find a suitable algebra B which is derived equivalent to A and for which the equivalence of Theorem 1 may be proved easily. Then, we prove that A is simply connected if and only if B is simply connected by establishing a correspondence between the Galois coverings of A and those of B. This correspondence is very similar to the one of [19] (and of [17]) since we shall compare the Galois coverings of A and those of $End_{\mathcal{D}^b(A)}(T)$ for some $T \in \mathcal{D}^b(A)$ (where $\mathcal{D}^b(A)$ denotes the derived category of bounded complexes of A-modules). Recall that, in [19], the suitable algebra B associated a piecewise hereditary algebra A of type Q was chosen to be the path algebra kQ. Here, we shall take for B a squid algebra (see [23]). Indeed, it was proved in [11] that a hereditary abelian k-linear category with tilting object and which is not derived equivalent to the module category of a hereditary algebra is derived equivalent to a squid algebra.

The text is organised as follows. In Section 1 we fix some notations. In Section 2, we construct the above correspondence. In Section 3, we prove the Theorem 1 for squid algebras. Finally, Section 4 is devoted to the proof of this theorem.

1 Notations

A k-category is a category whose collection $ob()$ of objects is a set, whose space of morphisms ${}_{y\,x}$ (or (x, y)) from x to y is a k-vector space for any $x, y \in ob()$, and whose composition of morphisms is k-bilinear. All functors between k-categories will be assumed to be k-linear. A basic connected finite dimensional k-algebra A will always be considered as a locally bounded k-category (see [5]) with set of objects a complete set $\{e_1, \ldots, e_n\}$ of pairwise orthogonal primitive idempotents, with space of morphisms from e_i to e_j equal to e_jAe_i and with composition of morphisms induced by the product of A.

Following [5], a (right) module over a locally bounded k-category is a k-linear covariant functor from ^{op} to the category of k-vector spaces. Such a module M is called finite dimensional if $\sum_{x\in ob()} dim_k\ M(x) < \infty$. In particular, for any $x \in ob()$, the indecomposable projective module $y \mapsto\ {}_{x\,y}$ will be denoted by ${}_{x\,?}$. The category of finite dimensional -modules will be denoted by $mod()$. The derived category of bounded complexes of -modules will be denoted by $\mathcal{D}^b()$ and Σ will denote the shift functor. Recall that if has finite global dimension, then $\mathcal{D}^b()$ is equivalent to the homotopy category of bounded complexes of finite dimensional projective -modules. The Auslander-Reiten translation (see [9]) on $\mathcal{D}^b()$ will be denoted by τ. Also, if $\mathcal{H}$ is a hereditary abelian category with tilting objects, then we shall write $\tau_{\mathcal{H}}$ for the Auslander-Reiten translation on $\mathcal{H}$ and on $\mathcal{D}^b(\mathcal{H})$.

For a reminder on Galois coverings, we refer the reader to [5] or [7]. A Galois covering $F\colon \ \to A$ will be called connected if and only if (and therefore A) is connected and locally bounded. Recall that if $F\colon \ \to A$ is a Galois covering with group G and with and A locally bounded, then F defines a triangle functor $F_\lambda\colon \mathcal{D}^b() \to \mathcal{D}^b(A)$ (see for example [19, Lem. 2.1]). Moreover, the group G acts on $\mathcal{D}^b()$ by triangle isomorphisms $(g, X) \in G \times \mathcal{D}^b() \mapsto\ {}^gX$. For this action, F_λ is G-invariant and for any $X, Y \in \mathcal{D}^b()$, the following maps induced by F_λ are linear isomorphisms:

$$\bigoplus_{g\in G} Hom_{\mathcal{D}^b()}({}^gX, Y) \xrightarrow{\sim} Hom_{\mathcal{D}^b(A)}(F_\lambda X, F_\lambda Y),\quad \bigoplus_{g\in G} Hom_{\mathcal{D}^b()}(X, {}^gY) \xrightarrow{\sim} Hom_{\mathcal{D}^b(A)}(F_\lambda X, F_\lambda Y)$$

For short, these properties on F_λ will be called the *covering properties of* F. Recall ([19, Lem. 4.1]) that F_λ verifies $\tau_A \circ F_\lambda \simeq F_\lambda \circ \tau$. An indecomposable object $X \in \mathcal{D}^b(A)$ is called of the first kind w.r.t. F if and only if $X \simeq F_\lambda \widetilde{X}$ for some $\widetilde{X} \in \mathcal{D}^b()$ (which is necessarily indecomposable). More generally, an object $X \in \mathcal{D}^b(A)$ is called of the first kind w.r.t. F if and only if X is the direct sum of indecomposable objects of the first kind w.r.t. F. Finally, for $T \in \mathcal{D}^b(A)$, we introduce two assertions depending on T and F and which will be used in this text:

(H_1) T is of the first kind w.r.t. F.

(H_2) for every indecomposable direct summand $X \in \mathcal{D}^b(A)$ of T, for any $\widetilde{X} \in \mathcal{D}^b()$ such that $F_\lambda\widetilde{X} \simeq X$ in $\mathcal{D}^b(A)$, and for any $g \in G\backslash\{1\}$, we have ${}^g\widetilde{X} \not\simeq \widetilde{X}$ in $\mathcal{D}^b()$.

For a reminder on tilting objects in hereditary abelian categories, we refer the reader to [12], for a reminder on cluster categories and on cluster tilting objects we refer the reader to [6]. The cluster category of a finite dimensional algebra A (resp. of a hereditary abelian category $\mathcal{H}$) will be denoted by C_A (resp. $C_{\mathcal{H}}$).

If is a triangulated category with shift functor Σ, we set $Ext^i(?,!) := Hom(?,\Sigma^i !)$. Also, if A is a finite dimensional algebra, we shall write Ext^i_A instead of $Ext^i_{\mathcal{D}^b(A)}$, for simplicity.

Finally, if $\mathcal{A}$ is a dg category, $Dif\ \mathcal{A}$ will denote the dg category of dg $\mathcal{A}$-modules, $\mathcal{D}(\mathcal{A})$ will denote the associated derived category. Recall that if $\mathcal{A}$ is a k-category considered as a dg category concentrated in degree 0, then $\mathcal{D}(\mathcal{A})$ is the usual derived category of (unbounded) complexes of $\mathcal{A}$-modules. If $\mathcal{B}$ is another dg category and if X is a dg $\mathcal{B}-\mathcal{A}$-bimodule, $?\underset{\mathcal{B}}{\otimes} X \colon Dif\ \mathcal{B} \to Dif\ \mathcal{A}$ will denote the tensor product dg functor and $?\overset{\mathbb{L}}{\underset{\mathcal{B}}{\otimes}} X \colon \mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathcal{A})$ will denote its left derived functor. For a reminder on dg categories, we refer the reader to [15].

2 Invariance of simple connectedness under tilting

Let A be a basic connected finite dimensional k-algebra. Let $\mathcal{D}^b(\mathcal{H}) \xrightarrow{\sim} \mathcal{D}^b(A)$ be a triangle equivalence where $\mathcal{H}$ is a hereditary abelian category with tilting objects. Finally, let $T \in \mathcal{D}^b(A)$ be a basic object such that:

1. T is a cluster tilting object of C_A
2. $Ext^i_A(T,T) = 0$ for any $i \neq 0$.

In this section, we shall compare the Galois coverings of A and those of $A' := End_{\mathcal{D}^b(A)}(T)$ in order to prove the following implication:

$$A' \text{ is simply connected } \Rightarrow A \text{ is simply connected} \qquad (\star\star)$$

Set $T = T_1 \bigoplus \ldots \bigoplus T_n \in \mathcal{D}^b(A)$ with $T_1, \ldots, T_n \in \mathcal{D}^b(A)$ indecomposables (where $n = rk(K_0(A))$). Recall that A' is a locally bounded k-category with set of objects $\{T_1, \ldots, T_n\}$, with space of morphisms from T_i to T_j equal to $Hom_{\mathcal{D}^b(A)}(T_i, T_j)$ and with composition of morphisms induced by the composition in $\mathcal{D}^b(A)$.

2.1 (Cluster) tilting objects of the first kind w.r.t. Galois coverings

In order to prove $(\star\star)$, we shall associate Galois coverings of A' to Galois coverings of A using a construction of [17, Sect. 2] and then use the characterisation [18, Cor. 4.5] of simple connectedness in terms of Galois coverings. In this purpose, the following lemma will be useful. Its proof is based on the work made in [19].

Lemma 2.1. *(see [19, Prop. 6.5, Prop. 6.8]) Let $F\colon \ \to A$ be a Galois covering with group G and with locally bounded. Then, (H_1) and (H_2) are satisfied for F and for any object of $\mathcal{D}^b(A)$ which is a cluster tilting object of C_A.*

Proof: For simplicity, we shall make no distinction between an object and its isomorphism class. Let $\subseteq \mathcal{D}^b(A)$ be the class of objects $R \in \mathcal{D}^b(A)$ which are isomorphic to the image of a cluster tilting object of $C_{\mathcal{H}}$ under the equivalence $\mathcal{D}^b(\mathcal{H}) \to \mathcal{D}^b(A)$. Hence, is the class of cluster tilting objects of C_A. In particular, it contains A. Let $\sim$ be the equivalence relation on generated by the following property: "*if $R, R' \in$ are such that $R = X \bigoplus \overline{R}$, $R' = Y \bigoplus \overline{R}$ with $X, Y \in \mathcal{D}^b(A)$ indecomposables and verifying at least one of the following properties:*

1. *$X \simeq (\tau_A \Sigma^{-1})^m Y$ for some $m \in \mathbb{Z}$,*
2. *there exists a triangle $X \to M \to Y \to \Sigma X$ of $\mathcal{D}^b(A)$ with $M \in add(\overline{R})$,*
3. *there exists a triangle $Y \to M \to X \to \Sigma Y$ of $\mathcal{D}^b(A)$ with $M \in add(\overline{R})$.*

then $R \sim R'$". Here, $add(R)$ denotes the full additive subcategory of $\mathcal{D}^b(A)$ closed under isomorphisms and generated by the indecomposable direct summands of R. Since any cluster tilting object of $C_{\mathcal{H}}$ is isomorphic (in $C_{\mathcal{H}}$) to a tilting object of $\mathcal{H}$ (see [6, Sect. 3]), since the Hasse diagram of tilting objects of

Ḫ is connected (see [14]), and since Ḑ^b(Ḫ) $\xrightarrow{\sim}$ Ḑ$^b(A)$ commutes with Σ and preserves Auslander-Reiten triangles, we infer that is an equivalence class for $\sim$. On the other hand, $F_\lambda\colon$ Ḑ^b() $\to$ Ḑ$^b(A)$ commutes with Σ and is compatible with τ and τ_A (*i.e.* $F_\lambda \circ \tau \simeq \tau_A \circ F_\lambda$, see [19, Lem. 4.1]). Therefore, using [19, Prop 6.5] and [19, Prop 6.8], we deduce that if $R \sim R'$, then the conclusion of the lemma holds for R and F if and only if it holds for R' and F. Since (H_1) and (H_2) are clearly satisfied for $T = A$, the lemma is proved. $\square$

2.2 Galois coverings of A' associated to Galois coverings of A

Now we can recall the construction of [17, Sect. 2] which associates Galois coverings of A' to Galois coverings of A. Fix $F\colon \ \to A$ a Galois covering with group G and with locally bounded. Assume that there exist $\widetilde{T}_1, \ldots, \widetilde{T}_n \in$ Ḑ^b() together with isomorphisms $\lambda_i\colon F_\lambda \widetilde{T}_i \xrightarrow{\sim} T_i$ in Ḑ$^b(A)$, for every i (see Lemma 2.1). Then, we define $'$ to be the following k-category:

1. the set of objects of $'$ is $\{\, {}^g\widetilde{T}_i \mid g \in G,\ i \in \{1, \ldots, n\}\}$.
2. ${}_{{}^h\widetilde{T}_j}{}'_{{}^g\widetilde{T}_i} := Hom_{Ḑ^b()}({}^g\widetilde{T}_i,\ {}^h\widetilde{T}_j)$ for any $g, h \in G$ and $i, j \in \{1, \ldots, n\}$.
3. the composition in $'$ is induced by the composition in Ḑ^b().

Hence, $'$ is the full subcategory of Ḑ^b() whose objects are the complexes ${}^g\widetilde{T}_i$. Moreover, we define a k-linear functor $F'\colon {}' \to A'$ as follows:

$$\begin{array}{rrcl} F'\colon & ' & \to & A' \\ & {}^g\widetilde{T}_i \in ob(') & \mapsto & T_i \in ob(A') \\ & u \in {}_{{}^h\widetilde{T}_j}{}'_{{}^g\widetilde{T}_i} & \mapsto & T_i \xrightarrow{\lambda_j \circ F_\lambda u \circ \lambda_i^{-1}} T_j \end{array}$$

The following lemma was proved in [17] in the case $T \in mod(A)$. However, the reader may easily check that the proof still works in our situation ($T \in$ Ḑ$^b(A)$):

Lemma 2.2. *(see [17, Rem. 2.1, Lem. 2.2]) The G-action on Ḑ^b() naturally defines a G-action on $'$. For this action, $F'\colon {}' \to A'$ is a Galois covering with group G and $'$ is a locally bounded k-category.*

Remark 2.3. *Since $Ext^m_A(T, T) = 0$ for any $m \neq 0$ and since F_λ has the covering property, we infer that $Ext^m({}^g\widetilde{T}_i,\ {}^h\widetilde{T}_j) = 0$ for any $g, h \in G$, $i, j \in \{1, \ldots, n\}$ and $m \in \mathbb{Z}\backslash\{0\}$.*

2.3 Connectedness of Galois coverings

Let us keep the notations of the preceding subsection. Since we are interested in *connected* Galois coverings, we need to check when $'$ is connected. In this purpose, we shall prove the following proposition.

Proposition 2.4. * and $'$ are derived equivalent. In particular, $'$ is connected if and only if is connected.*

We shall prove Proposition 2.4 in two steps: first we construct a fully faithful triangle functor $\Psi\colon$ Ḑ$^b(')$ $\to$ Ḑ^b() which maps the indecomposable projective $'$-module ${}_{{}^g\widetilde{T}_i}{}'?$ to an object of Ḑ^b() isomorphic to ${}^g\widetilde{T}_i \in$ Ḑ^b(). Then, we prove that this functor is dense.

Lemma 2.5. *There exists $\Psi\colon$ Ḑ$^b(')$ $\to$ Ḑ^b() a fully faithful triangle functor such that $\Psi({}_{{}^g\widetilde{T}_i}{}'?) \simeq {}^g\widetilde{T}_i$ in Ḑ^b(), for any $g \in G$, $i \in \{1, \ldots, n\}$. Moreover, Ψ has a right adjoint triangle functor Ḑ^b() $\to$ Ḑ$^b(')$.*

Proof: We may assume that ${}^g\widetilde{T}_i = {}^g\,\widetilde{T}_i^\bullet$ is a bounded complex of projective -modules, for any g, i.

- **A dg category Ḇ derived equivalent to $'$.** Denote by Ḇ the following dg category:

1. the set of objects is $\{\, {}^g\widetilde{T}_i \mid g \in G,\ i \in \{1, \ldots, n\}\}$,
2. Ḇ$^d({}^g\widetilde{T}_i,\ {}^h\widetilde{T}_j) := \Big\{(f_m\colon\ {}^g\widetilde{T}_i^m \to\ {}^h\widetilde{T}_j^{m+d})_{m\in\mathbb{Z}} \mid f_m$ is a morphism of -modules$\Big\}$,
3. the differential df of $f = (f_m)_{m\in\mathbb{Z}} \in$ Ḇ$^d({}^g\widetilde{T}_i,\ {}^h\widetilde{T}_j)$ is given by:

$$(df)_m = d^{m+d}_{{}^h\widetilde{T}_j} \circ f_m - (-1)^d f_{m+1} \circ d^m_{{}^g\widetilde{T}_i}$$

Since ${}^g\widetilde{T}_i$ is a bounded complex of projective -modules and thanks to Remark 2.3, there is an isomorphism of k-categories H^0Ḇ $\xrightarrow{\sim}$ $'$ extending the identity map on objects. To Ḇ is associated the sub dg

category $\tau_{\leqslant 0}\mathcal{B}$ with the same objects as $\mathcal{B}$ and such that $(\tau_{\leqslant 0}\mathcal{B})(X,Y)$ is truncated complex $\tau_{\leqslant 0}(\mathcal{B}(X,Y))$ for any X, Y. Thus, we have natural dg functors:

$$\mathcal{B} \leftarrow \tau_{\leqslant 0}\mathcal{B} \rightarrow H^0\mathcal{B}$$

Once again, by assumption on ${}^g\widetilde{T}_i$ and thanks to Remark 2.3, these functors induce isomorphisms of graded categories:

$$H^\bullet\mathcal{B} \xleftarrow{\sim} H^\bullet\tau_{\leqslant 0}\mathcal{B} \xrightarrow{\sim} H^0\mathcal{B} \tag{i}$$

On the other hand, $\tau_{\leqslant 0}\mathcal{B} \rightarrow \mathcal{B}$ (resp. $\tau_{\leqslant 0}\mathcal{B} \rightarrow H^0\mathcal{B}$) defines a dg $\tau_{\leqslant 0}\mathcal{B} - \mathcal{B}$-bimodule M (resp. a dg $\tau_{\leqslant 0}\mathcal{B} - H^0\mathcal{B}$-bimodule N) such that $M(X,Y) = \mathcal{B}(X,Y)$ for any $X \in ob(\mathcal{B})$ and $Y \in ob(\tau_{\leqslant 0}\mathcal{B})$ (resp. such that $N(X,Y) = H^0\mathcal{B}(X,Y)$ for any $X \in ob(H^0\mathcal{B})$ and $Y \in ob(\tau_{\leqslant 0}\mathcal{B})$). The bimodules M and N define triangle functors:

$$\mathcal{D}(\mathcal{B}) \xleftarrow{? \overset{\mathbb{L}}{\underset{\tau_{\leqslant 0}\mathcal{B}}{\otimes}} M} \mathcal{D}(\tau_{\leqslant 0}\mathcal{B}) \xrightarrow{? \overset{\mathbb{L}}{\underset{\tau_{\leqslant 0}\mathcal{B}}{\otimes}} N} \mathcal{D}(H^0\mathcal{B}) \tag{ii}$$

Using [15, 6.1] and the isomorphisms of (i), we infer that the above functors (ii) are triangle equivalences. Remark that since $H^0\mathcal{B}$ is concentrated in degree 0, the derived category $\mathcal{D}(H^0\mathcal{B})$ is exactly the derived category of complexes of $H^0\mathcal{B}$-modules. Finally, for any $X \in ob(\mathcal{B}) = ob(\tau_{\leqslant 0}\mathcal{B}) = ob(H^0(\mathcal{B}))$, we have ([16, 6.1]):

1. $X^\wedge \underset{\tau_{\leqslant 0}\mathcal{B}}{\otimes} M \simeq M(?, X) = X^\wedge$ in $Dif\ \mathcal{B}$,
2. $X^\wedge \underset{\tau_{\leqslant 0}\mathcal{B}}{\otimes} N \simeq N(?, X) = X^\wedge$ in $Dif\ H^0\mathcal{B}$,

so that:

1. $X^\wedge \overset{\mathbb{L}}{\underset{\mathcal{B}}{\otimes}} M \simeq X^\wedge$ in $\mathcal{D}(\tau_{\leqslant 0}\mathcal{B})$,
2. $X^\wedge \overset{\mathbb{L}}{\underset{H^0\mathcal{B}}{\otimes}} N \simeq X^\wedge$ in $\mathcal{D}(H^0\mathcal{B})$.

These isomorphisms together with the equivalences (ii) and the isomorphism $H^0(\mathcal{B}) \simeq {}'$ prove that there exists a triangle equivalence $\Phi\colon \mathcal{D}(') \xrightarrow{\sim} \mathcal{D}(\mathcal{B})$ which maps ${}_{{}^g\widetilde{T}_i}?'$ to an object of $\mathcal{D}(\mathcal{B})$ isomorphic to ${}^g\widetilde{T}_i^\wedge$, for any g, i.

• **The triangle functor** $? \overset{\mathbb{L}}{\underset{\mathcal{B}}{\otimes}} \widetilde{T}\colon \mathcal{D}(\mathcal{B}) \rightarrow \mathcal{D}()$. The complexes of -modules ${}^g\widetilde{T}_i$ naturally define a dg $\mathcal{B}$ − -bimodule $\widetilde{T}$ such that $\widetilde{T}(x, {}^g\widetilde{T}_i) = {}^g\widetilde{T}_i(x)$ for any ${}^g\widetilde{T}_i \in ob(\mathcal{B})$ and any $x \in ob()$. This bimodule defines a triangle functor:

$$? \overset{\mathbb{L}}{\underset{\mathcal{B}}{\otimes}} \widetilde{T}\colon \mathcal{D}(\mathcal{B}) \rightarrow \mathcal{D}()$$

Notice that [15, 6.1] implies that:

$$(\forall g, i) \quad {}^g\widetilde{T}_i^\wedge \overset{\mathbb{L}}{\underset{\mathcal{B}}{\otimes}} \widetilde{T} \simeq \widetilde{T}(?, {}^g\widetilde{T}_i) = {}^g\widetilde{T}_i \tag{iii}$$

Since ${}^g\widetilde{T}_i$ is a bounded complex of projective -modules, we infer (using [15, 6.2]), that $? \overset{\mathbb{L}}{\underset{\mathcal{B}}{\otimes}} \widetilde{T}$ admits a right adjoint triangle functor $\mathcal{D}() \rightarrow \mathcal{D}(\mathcal{B})$.

• **The triangle functor** $\Psi\colon \mathcal{D}^b(') \rightarrow \mathcal{D}^b()$ **and its right adjoint** $\Theta\colon \mathcal{D}^b() \rightarrow \mathcal{D}^b(')$. Let us set $\Psi :=? \overset{\mathbb{L}}{\underset{\mathcal{B}}{\otimes}} \widetilde{T} \circ \Phi\colon \mathcal{D}(') \rightarrow \mathcal{D}()$ and let us denote by $\Theta\colon \mathcal{D}() \rightarrow \mathcal{D}(')$ the composition of a quasi inverse of $\Phi\colon \mathcal{D}(') \xrightarrow{\sim} \mathcal{D}(\mathcal{B})$ with the right adjoint $\mathcal{D}() \rightarrow \mathcal{D}(\mathcal{B})$ of $? \overset{\mathbb{L}}{\underset{\mathcal{B}}{\otimes}} \widetilde{T}$. Thus, the pair (Ψ, Θ) is adjoint. Moreover, the construction of Φ and (iii) prove that:

$$(\forall g, i) \quad \Psi({}_{{}^g\widetilde{T}_i}?') \simeq {}^g\widetilde{T}_i \tag{iv}$$

This proves that Ψ maps $\mathcal{D}^b(')$ into $\mathcal{D}^b()$ and that it induces a triangle functor $\Psi\colon \mathcal{D}^b(') \rightarrow \mathcal{D}^b()$. Let us prove that Θ maps $\mathcal{D}^b()$ into $\mathcal{D}^b(')$. If $X \in \mathcal{D}^b()$, then:

1. $\bigoplus\limits_{g\in G} Hom_{\mathcal{D}^b()}(\Sigma^m\, {}^g\widetilde{T}_i, X) \simeq Hom_{\mathcal{D}^b(A)}(\Sigma^m T_i, F_\lambda X)$ is finite dimensional for any $i \in \{1,\ldots,n\}$ and $m \in \mathbb{Z}$ (recall that $F_\lambda \colon \mathcal{D}^b() \to \mathcal{D}^b(A)$ has the covering property).
2. there exists $m_0 \in \mathbb{N}$ such that $Hom_{\mathcal{D}^b()}(\Sigma^m\, {}^g\widetilde{T}_i, X) = 0$ for any $g \in G$, $i \in \{1,\ldots,n\}$ and $m \in \mathbb{Z}$ such that $|m| \geqslant m_0$.

These two properties imply that $\sum\limits_{g,i,m} dim_k\ \mathcal{D}()(\Sigma^m\, {}^g\widetilde{T}_i, X) < \infty$. Using the fact that (Ψ,Θ) is adjoint and using (iv), we deduce that $\sum\limits_{g,i,m} dim_k\ \mathcal{D}(')(\Sigma^m\ {}_{{}^g\widetilde{T}_i}{}'?, \Theta(X)) < \infty$. This proves that $\Theta(X) \in \mathcal{D}^b(')$. Therefore, Θ induces a triangle functor $\Theta \colon \mathcal{D}^b() \to \mathcal{D}^b(')$ such that the pair (Ψ,Θ) is adjoint:

$$\begin{tikzcd} \mathcal{D}^b(') \arrow[d, bend right, "\Psi"'] \\ \mathcal{D}^b() \arrow[u, bend right, "\Theta"'] \end{tikzcd}$$

• $\Psi \colon \mathcal{D}^b(') \to \mathcal{D}^b()$ **is fully faithful**. For short, if $X, Y \in \mathcal{D}^b(')$, we shall write $\Psi_{X,Y}$ for the mapping $Hom_{\mathcal{D}^b(')}(X,Y) \to Hom_{\mathcal{D}^b()}(\Psi(X),\Psi(Y))$ induced by Ψ. Let $g,h \in G$ and $i,j \in \{1,\ldots,n\}$. Then $Hom_{\mathcal{D}^b(')}(\ {}_{{}^g\widetilde{T}_i}{}'?,\ {}_{{}^h\widetilde{T}_j}{}'?) = Hom_{\mathcal{D}^b()}(\ {}^g\widetilde{T}_i,\ {}^h\widetilde{T}_j)$. Moreover, we have (iv) $\Psi(\ {}_{{}^g\widetilde{T}_i}{}'?) \simeq\ {}^g\widetilde{T}_i$ and $\Psi(\ {}_{{}^h\widetilde{T}_j}{}'?) \simeq\ {}^h\widetilde{T}_j$, and with these identifications, $\Psi_{{}_{{}^g\widetilde{T}_i}{}'?,\,{}_{{}^h\widetilde{T}_j}{}'?}$ is the identity mapping. On the other hand, if $m \in \mathbb{Z}\backslash\{0\}$, then $\Psi_{\Sigma^m(\,{}_{{}^g\widetilde{T}_i}{}'?),\,{}_{{}^h\widetilde{T}_j}{}'?}$ is an isomorphism because the involved morphisms spaces are trivial. Hence, $\Psi_{\Sigma^m X,Y}$ is an isomorphism for any $m \in \mathbb{Z}$ and any projective $'$-modules X, Y. This shows that Ψ is fully faithful. $\square$

Lemma 2.6. *The set* $\{\ \tau^l\Sigma^m\, {}^g\widetilde{T}_i \mid m,l \in \mathbb{Z},\ g \in G,\ i \in \{1,\ldots,n\}\}$ *generates* $\mathcal{D}^b()$ *as a triangulated category. Therefore, the functor* Ψ *of Lemma 2.6 is dense.*

Proof: Let and $\sim$ be as in the proof of Lemma 2.1. For any $R = R_1 \bigoplus \ldots \bigoplus R_n \in$, fix $\widetilde{R}_1,\ldots,\widetilde{R}_n \in \mathcal{D}^b()$ indecomposables such that $F_\lambda\widetilde{R}_i \simeq R_i$ for every i (see Lemma 2.1). Then denote by $< R >$ for the full triangulated subcategory of $\mathcal{D}^b(')$ generated by $\{\ \tau^l\Sigma^m\, {}^g\widetilde{R}_i \mid m,l \in \mathbb{Z},\ g \in G,\ i \in \{1,\ldots,n\}\}$. Remark that $< R >$ does not depend on the choice of $\widetilde{R}_1,\ldots,\widetilde{R}_n$ because if $\widetilde{R}'_i \in \mathcal{D}^b()$ verifies $F_\lambda\widetilde{R}'_i \simeq F_\lambda\widetilde{R}_i \simeq R_i$, then there exists some $g \in G$ such that ${}^g\widetilde{R}_i \simeq \widetilde{R}'_i$ (see for example the proof of [19, Lem. 5.3]). Since $< A >$ contains all the indecomposable projective -modules up to isomorphism, we infer that $< A >= \mathcal{D}^b()$. On the other hand, the second assertion of [19, Prop. 6.5] proves that if $R \sim R'$, then $< R >=< R' >$. Since is an equivalence class for $\sim$ (see the proof of Lemma 2.1), we infer that $< R >=< A >= \mathcal{D}^b()$ for any $R \in$. This proves the first assertion of the lemma.

In order to prove the second assertion of the lemma, it suffices to prove that the image of $\Psi \colon \mathcal{D}^b(') \to \mathcal{D}^b()$ contains $< T >$. By construction, this image contains $\{\,{}^g\widetilde{T}_i \mid g \in G,\ i \in \{1,\ldots,n\}\}$. On the other hand, $\Psi \colon \mathcal{D}^b(') \to \mathcal{D}^b()$ is fully faithful and admits a right adjoint, so it preserves Auslander-Reiten sequences. In particular, we have $\tau \circ \Psi \simeq \Psi \circ \tau_{'}$. This proves that the image of Ψ contains $< T >= \mathcal{D}^b()$. $\square$

Using Lemma 2.5 and Lemma 2.6, the proof of Proposition 2.4 is immediate. Now, we are able to prove the announced implication $(\star\star)$:

Proposition 2.7. *For any group* G*, there exists a connected Galois covering with group* G *of* A' *if there exists a connected Galois covering with group* G *of* A'*. Consequently:*

$$A' = End_{\mathcal{D}^b(A)}(T) \text{ is simply connected } \Rightarrow\ A \text{ is simply connected}$$

Proof: Let us assume that A' is simply connected. If $F\colon \ \to A$ is a connected Galois covering with group G, then Lemma 2.2 and Proposition 2.4 show that there exists $F' \colon {}' \to End_{\mathcal{D}^b(A)}(T)$ a connected Galois covering with group G. Since A' is simply connected, we infer (see [18, Cor. 4.5]) that G is necessary the trivial group. Hence (*loc. cit.*) A is simply connected. $\square$

3 Hochschild cohomology and simple connectedness of squid algebras

We refer the reader to [23] for more details on squid algebras. A squid algebra over an algebraically closed field k is defined by the following data: an integer $t \geqslant 2$, a sequence $p = (p_1, \ldots, p_t)$ of non negative integers and a sequence $\tau = (\tau_3, \ldots, \tau_t)$ of pairwise distinct non zero elements of k. With this data, the squid algebra $S(t, p, \tau)$ is the k-algebra kQ/I where Q is the following quiver:

$$\begin{tikzcd}
 & & (1,1) \arrow[r] & \cdots \arrow[r] & (1,p_1) \\
{} \arrow[r, bend left, "a_1"] \arrow[r, bend right, "a_2"'] & {} \arrow[ur, "b_1"] \arrow[r, "b_2"] \arrow[ddr, "b_t"'] & (2,1) \arrow[r] & \cdots \arrow[r] & (2,p_2) \\
 & & \vdots & & \vdots \\
 & & (t,1) \arrow[r] & \cdots \arrow[r] & (t,p_t)
\end{tikzcd}$$

and I is the ideal generated by the following relations:

$$b_1 a_1 = b_2 a_2 = 0, \quad b_i a_2 = \tau_i \, b_i a_1 \ \text{ for } i = 3, \ldots, t$$

Using Happel's long exact sequence ([10]), one can compute $HH^1(S(t, p, \tau))$:

$$\dim_k \mathrm{HH}^1(S(t,p,\tau)) = \begin{cases} 1 & \text{if } t = 2 \\ 0 & \text{if } t \geqslant 3 \end{cases}$$

On the other hand, one checks easily that if $t = 2$ then the fundamental group $\pi_1(Q, I)$ of the above presentation of $S(t, p, \tau)$ is isomorphic to $\mathbb{Z}$ (see [20]), whereas $S(t, p, \tau)$ is simply connected if $t \geqslant 3$. These considerations imply the following proposition.

Proposition 3.1. *Let A be a squid algebra. Then A is simply connected if and only if $HH^1(A) = 0$.*

4 Proof of Theorem 1

Now we can prove Theorem 1. Let A be quasitilted *i.e.* $A = End_{\mathcal{H}}(X)^{op}$ where $\mathcal{H}$ is hereditary abelian and where $X \in \mathcal{H}$ is basic tilting. If $\mathcal{H}$ is derived equivalent to $mod(kQ)$ for some quiver Q, then the conclusion of the theorem follows from [19, Cor. 2]. Otherwise, there exists $\mathcal{H}'$ a hereditary abelian category, there exists a triangle equivalence $\mathcal{D}^b(\mathcal{H}) \xrightarrow{\sim} \mathcal{D}^b(\mathcal{H}')$ and there exists $Y \in \mathcal{H}'$ basic tilting such that $End_{\mathcal{H}'}(Y)^{op}$ is a squid algebra (see [11, Prop. 2.1, Thm. 2.6]). Set $B := End_{\mathcal{H}'}(Y)^{op}$. Then:

1. there exist triangle equivalences $\mathcal{D}^b(\mathcal{H}) \xrightarrow{\sim} \mathcal{D}^b(A)$ and $\mathcal{D}^b(\mathcal{H}') \xrightarrow{\sim} \mathcal{D}^b(B)$ mapping X and Y to A and B respectively (thanks to [12, Thm. 3.3, Thm 4.3]).
2. if $T \in \mathcal{D}^b(A)$ denotes the image of $Y \in \mathcal{H}'$ under the equivalence $\mathcal{D}^b(\mathcal{H}') \xleftarrow{\sim} \mathcal{D}^b(\mathcal{H}) \xrightarrow{\sim} \mathcal{D}^b(A)$, then: (i) $Ext^i_A(T, T) = 0$ for every $i \neq 0$, (ii) T is a cluster tilting object of C_A, (iii) $End_{\mathcal{D}^b(A)}(T) \simeq End_{\mathcal{H}'}(Y)$.
3. if $T' \in \mathcal{D}^b(B)$ denotes the image of $X \in \mathcal{H}$ under the equivalence $\mathcal{D}^b(\mathcal{H}) \xrightarrow{\sim} \mathcal{D}^b(\mathcal{H}') \xrightarrow{\sim} \mathcal{D}^b(B)$, then: (iv) $Ext^i_B(T', T') = 0$ for every $i \neq 0$, (v) T' is a cluster tilting object of C_B, (vi) $End_{\mathcal{D}^b(B)}(T) \simeq End_{\mathcal{H}}(X)$.

Now, Proposition 2.7 applied A and T and to B and T' proves that A is simply connected if and only if B is simply connected (recall that A is simply connected if and only if A^{op} is simply connected, see for example the proof of [19, Thm. 3]). Since $\mathcal{D}^b(A)$ and $\mathcal{D}^b(B)$ are triangle equivalent, we have $HH^1(A) \simeq HH^1(B)$ as k-vector spaces (see [16]). Finally, Proposition 3.1 applied to B proves that A is simply connected if and only if $HH^1(A) = 0$. $\square$

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